The Unimodular Determinant Spectrum Problem

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What are the possible determinants of ± 1 matrices?

Order 2	$\{0, \pm 2\}$
Order 3	$\{0, \pm 4\}$
Order 4	$\{0, \pm 8, \pm 16\}$
Order 5	$\{0, \pm 16, \pm 32, \pm 48\}$

The Determinant Spectrum Problem

Definition

The order n ± 1 **determinant spectrum** is the set of values taken by $\frac{|\det(A)|}{2^{n-1}}$ as A ranges over all possible ± 1 matrices of order n.

What is the order n spectra for each $n \in \mathbb{N}$?

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- Dates back to James Sylvester in 19th century
- Hadamard matrices satisfy upper bound $(n = 1, 2, 4k)$
- Solved for sizes up to size $n = 11$ and for $n = 13$.
- Conjectures formulated for sizes up to $n = 22$

Spectra

For each $n \in \mathbb{N}$ let D_n denote the order n spectrum. Given our choice for scaled determinant values, namely $\frac{|\det(A)|}{2^{n-1}}$, the initial spectra are:

- $D_2 = \{0, 1\}$
- $D_3 = \{0, 1\}$
- $D_4 = \{0, 1, 2\}$
- $D_5 = \{0, 1, 2, 3\}$
- $D_6 = \{0, 1, 2, 3, 4, 5\}$
- $D_7 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

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- All known spectra larger than $n = 7$ contain gaps.

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- $D_2 = \{0, \sqrt{2}, 2\}$
- $D_3 = \{0, 2, 2\sqrt{2}, 4, 2\sqrt{5}\}$
- D_4 has 14 values

"...probably completely intractable."
-Dr. R. Craigen

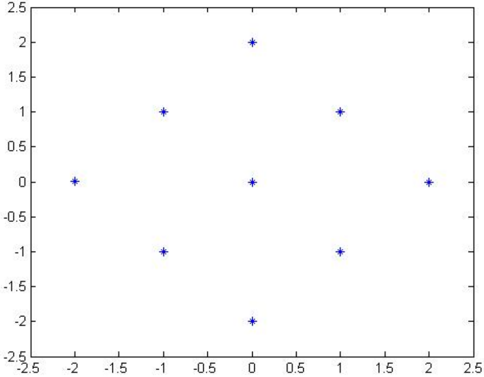
The Determinant Spectrum Problem For Quartic Root of Unity Matrices

Definition

The order n quartic root of unity **determinant spectrum** is the set of values taken by $\det(A)$ (NOT $\frac{|\det(A)|}{2^{n-1}}$) as A ranges over all possible order n quartic root of unity matrices matrices.

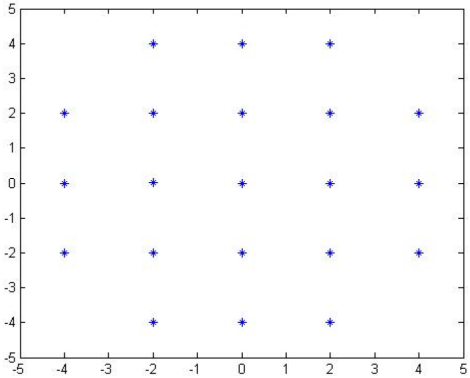
What is the order n quartic root of unity determinant spectrum for each $n \in \mathbb{N}$?

Order 2 Spectrum



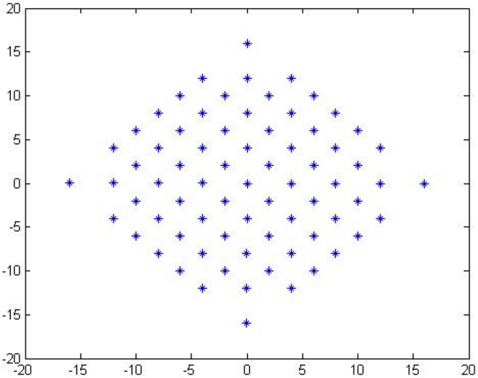
$$\{0, (1+i)i^k, 2i^k | k \in \mathbb{Z}_4\}$$

Order 3 Spectrum



$$\{0, 2i^k, (2+2i)i^k, 4i^k, (2 \pm 4i)i^k | k \in \mathbb{Z}_4\}$$

Order 4 Spectrum

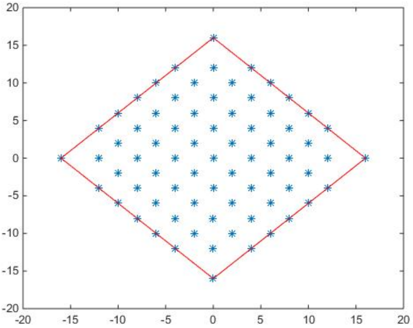


Theorem (*)

The order 4 spectrum consists of

$$\{a+bi \in \mathbb{Z}[i] | a, b \in 2\mathbb{Z}, a \equiv b \mod 4, |a|+|b| \leq 16\}$$

except those of the form $i^k(14 \pm 2i)$ *for some power of* k*.*

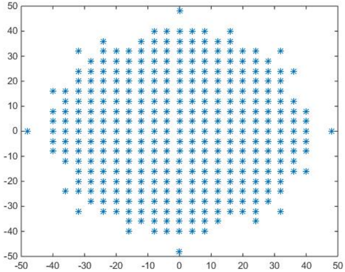


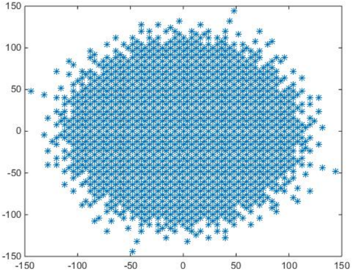
Proof Outline

- Spectrum closed under multiplication by i^k
- Real and imaginary even parts congruent modulo 4
- Chió's method reduced 4-by-4 determinant to a 3-by-3
- Analyze cases according to matrix entries

Future Work

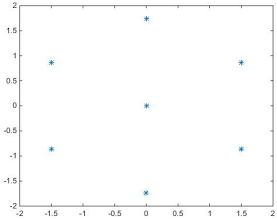
Orders 5 and 6

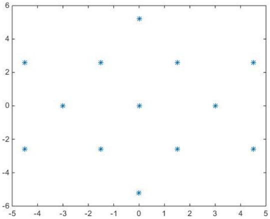


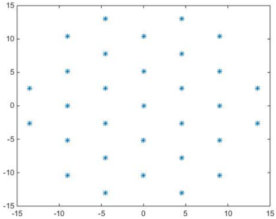


Future Work

3rd Roots of Unity







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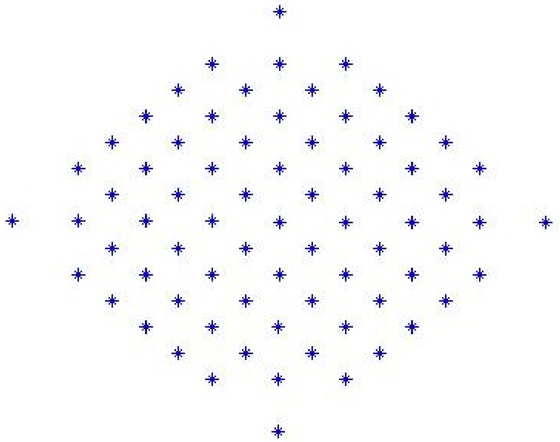
QUESTIONS?



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